§-1. instanton number

K : cpt, simple Lie group $\quad (G = K_{\mathbb{C}}$: cpxification$)$
simply-connected, rank $= l$

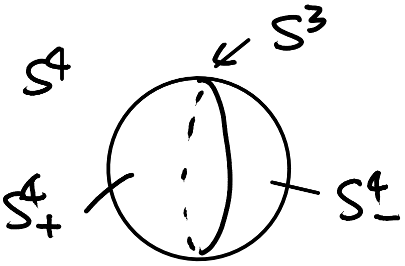
We are interested in K-instantons on S^4
$=$ anti-self-dual connections on a principal K-b'dle P over S^4

framed moduli space : $A \sim \gamma A$
$\uparrow$
gauge transform. $\gamma(\infty) = \mathrm{id}$
$S^4 = \mathbb{R}^4 \cup \infty$

algebro-geometric approach : $\mathcal{E}$
framed holomorphic G-b'dles over $\mathbb{P}^2 = \mathbb{C}^2 \cup l_\infty$
$\varphi : \mathcal{E}|_{l_\infty} \cong l_\infty \times G$

<u>instanton numbers</u>

K-b'dles P on S^4 are classified by
<u>integers</u> $\quad \pi_3(K) \cong \mathbb{Z}$
$\uparrow$ called instanton numbers



$P = S^4_+ \times G \cup S^4_- \times G$
$\gamma : S^3 \to G$

$\pi_3(K) \cong \mathbb{Z} \; (\cong \pi_3(S^3))$
is realised by

$$\begin{array}{ccc} S^3 & & \\ \wr\wr & & \\ SU(2) & \longrightarrow & K \\ \cap & & \cap \\ SL(2) & \longrightarrow & G \\ f & \longmapsto & f_\theta \end{array}$$

θ : highest root

It also means that any K-b'dle P can be topologically reduced to an $SU(2)$-b'dle with the same instanton number

n : instanton number

$\mathrm{Bun}_G(n)$: framed moduli space of K-instantons on S^4 $\;/\!/\; \mathbb{R}^4 \cup \infty$

$=$ framed moduli space of holomorphic G-b'dles on $\mathbb{P}^2 = \mathbb{C}^2 \cup \ell_\infty$

The dimension of instanton moduli spaces is computed by Atiyah-Singer index theorem

$$\int_{S^4} \underset{\substack{\| \\ [\mathrm{tr}(\mathrm{ad} F_A)^2]}}{p_1(\mathrm{ad} P)}\, \mathrm{ch}(S_+)\hat{A}$$

$\uparrow$ spinor b'dle

Atiyah-Hitchin-Singer

F_A: curvature of a connection

$$P = P_{SU(2)} \times_{SU(2)} K$$

B_G: Killing form

$$\therefore\; p_1(\mathrm{ad} P) = p_1(\mathrm{ad} P_{SU(2)}) \times \frac{B_G}{B_{SU(2)}}$$

$$= p_1(\mathrm{ad} P_{SU(2)}) \times \frac{B^\vee_{SU(2)}(\theta,\theta)}{B^\vee_G(\theta,\theta)} = p_1(\mathrm{ad} P_{SU(2)}) \times \frac{h^\vee}{2}$$

dual Coxeter number

$\dim_{\mathbb{C}}$ for $SU(2) = 4n$ $\qquad \therefore \dim_{\mathbb{C}}$ for $K = 2nh^\vee$

<u>Rem.</u> α : root $\quad SU(2) \xrightarrow{\rho_\alpha} K$ corresponding embedding

$$P = P_{SU(2)} \times_{\rho_\alpha} K \Rightarrow n(P) = n(P_{SU(2)}) \times \frac{B^\vee_G(\theta,\theta)}{B^\vee_G(\alpha,\alpha)}$$

$1, \underbrace{2 \text{ or } 3}$

long root $\quad$ short root

§0. Statement

Uhlenbeck space $\mathcal{U}_G(n) \supset \mathrm{Bun}_G(n)$
└ partial compactification

properties

- $G = SL(r) \Rightarrow \mathcal{U}_G(n) = M_0(n,r)$
  $= \mu^{-1}(0) /\!/ GL(n)$ in ADHM
- [BFG] $\mathcal{U}_G(n)$: affine algebraic variety
  0301176 have an action of $G \times GL(2)$
- $\mathcal{U}_G(n)$ has the canonical stratification

$$\mathcal{U}_G(n) = \coprod_{m+|\lambda|=n} \mathrm{Bun}_G(m) \times \underbrace{S_\lambda \mathbb{A}^2}_{\shortparallel} \qquad \lambda = \lambda_1 \geqq \lambda_2 \geqq \cdots$$

$$\{ \textstyle\sum \lambda_i x_i \mid x_i : \text{distinct} \}$$

- $n = 1 \quad \mathcal{U}_G(1) = \mathbb{A}^2 \times \overline{\text{minimal nilpotent orbit}}$

- $$\begin{tikzcd} \mathrm{Bun}_G(m) \times S^{n-m}\mathbb{A}^2 \arrow[r, hook] \arrow[d, phantom, sloped, "\subset"] & \mathcal{U}_G(n) \\ \mathcal{U}_G(m) \times S^{n-m}\mathbb{A}^2 \arrow[ur, "\exists \text{ finite morphism}"'] & \end{tikzcd}$$

  This is not bijective, as we have choices to separate singularities.

- $G = SL(r)$
  $\Rightarrow \mathcal{U}_G(n) = M_0(n,r)$ has a symplectic resolution
  $M(n,r) \longrightarrow M_0(n,r)$

"Th". But $G \neq SL(r)$, $\mathcal{U}_G(n)$ does <u>not</u> have a symp. resol.

"proof" $n=1 \Rightarrow$ Fu : 0205048
$n>1 \Rightarrow$ slice argument ??

Now consider
$$\mathbb{T} = T \times T^2 \subset G \times GL(2) \quad \text{max. torus}$$

$IH^*_{\mathbb{T}}(\mathcal{U}_G(n))$, $IH^*_{\mathbb{T},c}(\mathcal{U}_G(n))$ intersection cohomology
$\leftarrow$ cpt support

(degree is shifted by $2nh^\vee = \dim_{\mathbb{C}} \mathcal{U}_G(n)$)

module over $H^*_{\mathbb{T}}(pt) = S(\operatorname{Lie}\mathbb{T}^*)$
$$\overset{\parallel}{A_{\mathbb{T}}} \qquad = \mathbb{Q}[\underset{\substack{\parallel \\ (a_1,\cdots,a_\ell)}}{\vec{a}}, \varepsilon_1, \varepsilon_2]$$
$(a_1,\cdots,a_\ell)$ coord. of $\operatorname{Lie} T$

<u>Prop</u>. $IH^*_{\mathbb{T}(c)}(\mathcal{U}_G(n))$: free over $A_T = \mathbb{Q}[\varepsilon_1, \varepsilon_2, \vec{a}]$

$\therefore$ $IH^*(\mathcal{U}_G(n)) \otimes H^*_{\mathbb{T}}(pt) \Rightarrow IH^*_{\mathbb{T}}(\mathcal{U}_G(n))$ degenerates
as $IH^{odd}(\mathcal{U}_G(n)) = 0$ //
$\uparrow$ [BFG]

Poincaré duality : $IH^*_{\mathbb{T}}(\mathcal{U}_G(n)) \otimes IH^*_{\mathbb{T},c}(\mathcal{U}_G(n)) \longrightarrow H^*_{\mathbb{T}}(pt)$
$\int_{\mathcal{U}_G(n)} \cdot \cup \cdot$ is a perfect pairing.

& $IH^*_{\mathbb{T},c}(\mathcal{U}_G(n)) \longrightarrow IH^*_{\mathbb{T}}(\mathcal{U}_G(n))$ natural hom.

'Th' Suppose G is of type ADE

(1) $\bigoplus_{n=0}^{\infty} IH^*_{\mathbb{T},c}(\mathcal{U}_G(n))/\text{torsion} \cong {}_{A_T}M(\vec{a})$

the A_T-form of the universal Verma module $M(\vec{a})$ ← h.w.
of the W-algebra $W_k(\mathfrak{g})$, where $k + h^\vee = -\frac{\varepsilon_2}{\varepsilon_1}$

$\bigoplus IH^*_{\mathbb{T}}(\mathcal{U}_G(n))/\text{torsion} \cong {}^{A_T}M(\vec{a})$ its dual module.
$(\ ,\ ) \leftrightarrow$ Shapovalov form

(2) $[\mathcal{U}_G(n)] \in \bigoplus IH^*_{\mathbb{T}}(\mathcal{U}_G(n))$ satisfies the Whittaker vector property.

$p = 1, \cdots, l$
$m = 1, 2, \cdots$

$$W_m^{(p)}[\mathcal{U}_G(n)] = \begin{cases} \pm[\mathcal{U}_G(n-1)] & \text{if } m=1,\ p=l \\ 0 & \text{otherwise} \end{cases}$$

at this moment

<u>Remark</u>

(1) For type A, it is a slight refinement of integrality [Schiffmann-Vasserot], [Maulik-Okounkov]
But our proof depends on [SV], [MO].

(2) The definition of $A_T M(\vec{a})$ will <u>not</u> be given
It is a module of ${}_A W(\mathfrak{g}) = \mathbb{Q}[\varepsilon_1, \varepsilon_2]$-form of $W_k(\mathfrak{g})$
$= A\text{-span}\langle W^{(p)} \mid p = 1, \cdots, l \rangle$

<u>e.g.</u> Virasoro $[\tilde{L}_m, \tilde{L}_n] = (m-n)\varepsilon_1\varepsilon_2 \tilde{L}_{m+n}$
$+ \left((\varepsilon_1\varepsilon_2)^2 + 6\varepsilon_1\varepsilon_2(\varepsilon_1+\varepsilon_2)^2\right)\delta_{m,-n}\frac{m^3-m}{12}$

(3) G: nonsimply-laced
$\Rightarrow W_k(\mathfrak{g})$ should be replaced by the BRST reduction of the twisted affine Lie alg. $\mathfrak{g}^L_{\text{aff}}$
= Langlands dual of $\mathfrak{g}_{\text{aff}}$.

(4) ?? $\mathcal{W}_k(\mathfrak{g})$ for nonsimplylaced $\mathfrak{g}$

$\leftarrow$ instanton twisted by $\mathbb{Z}_2$ or $\mathbb{Z}_3$

(5) $\mathcal{W}_k(\mathfrak{g}, x)$

a nilpotent orbit of $\mathfrak{g}$

$x = x_{reg}$ in 'Th'

Suppose x : regular in levi $\mathfrak{l}$ $\quad$ P : corr. parabolic

$\overset{?}{\Rightarrow}$ Conj. The same is true for bdle with parabolic P str. along $x=0$

More general x, we do not even have a conjectural statement. Tachikawa's result suggests a local system ?

More explanation on the statement (1)

Choose $\mathcal{C} \subset \mathrm{Lie}\, T$: Weyl chamber

$$\Rightarrow \exists\, \Phi^{\mathcal{C}} : \bigoplus_n IH^*_{\mathbb{T},c}(\mathcal{U}_G(n)) \longrightarrow \mathbb{A}_T F$$

$\mathbb{A}_T F$: the integral form of the Fock module of $\mathrm{Heis} = \widehat{\mathfrak{t}[z, z^{-1}]}$

$$[\tilde{P}^i_m, \tilde{P}^j_n] = m\delta_{m,-n}(\alpha_i, \alpha_j)\varepsilon_1\varepsilon_2$$

This is the integral version of Feigin-Frenkel $W \subset \mathrm{Heis}$.

The $\mathbb{A}_T$-form of universal Verma of W $= \bigcap_i$ The $\mathbb{A}_T$-form of universal form of Vir_i for each i

- $\Phi^{\mathfrak{C}}$ is $\cong$ over $\mathrm{Frac}(\mathbb{Q}[\vec{a}])$
- $\Phi^{\mathfrak{C}} \circ (\Phi^{\mathfrak{C}'})^{-1}$ = R-matrix with spectral param.

  YB eqn. $\Rightarrow$ depending on w : $\mathfrak{C}' = w\mathfrak{C}$

◎ braid rel. is rank 2 reduction

$\therefore$ ADE $\longrightarrow$ reduction to $A_2 \leftarrow$ [MO]

BCG $\longrightarrow$ $B_2 . G_2$ are necessary.

Not known yet.

The key of our proof is the geometric construction of $\Phi^{\mathfrak{C}}$ = Feigin-Frenkel embedding.